Roles of convection, pressure, and dissipation in three-dimensional turbulence

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We investigated the roles of the convection, pressure, and dissipation terms in the equation for the longitudinal velocity increment, with the help of the DNS data on 1024^3. The pressure screens the convection growing as the intensity of fluctuation increases. The dissipation term is found to make no direct contribution to the fourth order structure function; the structure functions of order not less than 5 are affected by the dissipative structure as far as the scaling is concerned. A reason is also given for the observation by Stolovitzky *et al.* [Phys. Rev. E **48**, R3217 (1993)] that the scaling of the $(2m)$th order structure function is more similar to that of the $(2m-1)$th order structure function than that of $(2m+1)$th order structure function.

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I. INTRODUCTION

The scaling of the longitudinal structure function is one of the main subjects in turbulence [1–3]. It is now well established [4–8] that the scaling deviates from the original Kolmogorov scaling, i.e., K41 [9]. What is responsible for the deviation? The spatially nonuniform energy dissipation rate [10,11] has been thought of as a main candidate for it since Landau's comment. Recently Yakhot [12] proposed the mean field approximation, which was based on the neglect of the pressure term and the dissipation term, and the addition of these terms in a perturbative way. Kurien and Sreenivasan [13] investigated the validity of his various equations using the data in the atmospheric boundary turbulence. In this context it is interesting to know whether and how the dissipative structure and the pressure gradient affect the scaling in turbulence. Such a study is also useful for modeling turbulence in LES.

Currently the anomalous scaling has been addressed for the passive scalar field advected by the delta-correlated velocity field [14–21]. In this issue the advection and dissipation terms are combined into a single operator of second derivative, and a homogeneous solution of the operator is associated with the anomalous scaling of the scalar structure function [17,21]. The three-dimensional turbulence is, however, much more complicated for the following reasons. (i) Since the convection term is nonlinear, the dissipation term cannot be combined with the convection term. (ii) There is a nonlocal pressure term, which is the superposition of the divergence of the convection term from afar. The present paper aims to theoretically investigate roles of the convection, pressure, and dissipation terms in the equation for the longitudinal velocity increment, with the assistance of the DNS on 1024^3 meshes.

We will employ two methods. The first is to compute the conditional averages of the three fundamental terms, i.e., convection, pressure, and dissipation terms, with a value of the longitudinal velocity difference w_1 being fixed. Since the conditional averages are a function of w_1, the averaged quantities reveal the amplitude-dependent effects of those terms. The second is to investigate the equation for a structure function of arbitrary order, which consists only of the convection, pressure, and dissipation terms in the inertial region. We check how three terms satisfy the equation in a balanced way.

The findings in this work are as follows. We start with the equation for the velocity increment in physical space, from which the equation for the structure function is derived; it contains only the convection, pressure, and dissipation terms in the inertial region. In order to know the roles of the convection and the pressure we compute the conditional averages of the convection and pressure terms with a fixed value w_1 of the longitudinal velocity difference from the DNS data. The sign of the conditional average of the pressure term is found to be opposite to that of the convection term, so that the pressure screens the convection partly. The relative magnitude of both averages is estimated; the convection term is predominant over the pressure one at $w_1<0$, while the latter is larger than the former at $w_1>0$.

The relation between the inertial term (the convection term plus the pressure term) and the dissipation term is examined based on the equation for the structure function. The dissipative term is negligible as compared with the inertial term in the inertial region of the fourth order structure function, while it becomes comparable and balances with the inertial term for the structure functions of order equal to or higher than 5. The pressure is found to screen the convection increasingly as the order of the structure function grows.

Next we turn to the correlation I_n of the longitudinal dis-

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sipation field with the nth power of the longitudinal velocity difference. For $n \geqslant 2$, I_n scales with the exponent less than the K41 value $n/3$ in the inertial region, while I_1 decreases with r, unlike the K41 expectation $r^{1/3}$. We also show that the scaling exponent of I_{2n-1} is closer to that of I_{2n-2} than that of I_{2n}. This finding is consistent with the observation by Stolovitzky *et al.* [22] that the scaling of $\langle w_1^{2m} \rangle$ is closer to that of $\langle w_1^{2m-1} \rangle$ than that of $\langle w_1^{2m+1} \rangle$. The correlation J_n of the longitudinal dissipation rate with the nth power of the absolute value of the longitudinal velocity increment is also investigated; J_n behaves orderly with n in contrast to I_n, indicating the scaling exponent of the structure function associated with the absolute value of the longitudinal velocity increment increases with n orderly in the inertial region in accordance with our previous result [8].

The present paper is organized in the following way. In Sec. II we prepare the fundamental equation for the velocity increment in physical space, from which the equation for the structure function is derived. Section III is devoted to the analysis of the conditional average of the convective and pressure terms with a fixed value w_1 of the longitudinal velocity difference. First, those averages are computed from the DNS data, and then their features are theoretically interpreted. The relative order of magnitude of the both conditional averages is estimated in Sec. IV. In Sec. V we investigate the relation between the inertial term and the dissipation term based on the equation for the structure function. Section VI is devoted to the evaluation of the correlation of the longitudinal dissipation field with an arbitrary power of the longitudinal velocity difference. We also consider the correlation of the dissipation rate with an arbitrary order of the absolute value of the longitudinal velocity increments. In Sec. VII we discuss the results obtained in the present paper in comparison with the mean-field-approximated results [12,13]. Section VIII is devoted to the discussion as to the anomalous scaling of longitudinal velocity increments.

II. EQUATION FOR VELOCITY DIFFERENCE

Let us introduce the velocity increment at two points $\mathbf{x}_1$ and $\mathbf{x}_2$ as

$$w_i(\mathbf{x}_1,\mathbf{x}_2)=u_i(\mathbf{x}_2)-u_i(\mathbf{x}_1). \tag{2.1}$$

The equation for w_i is obtained by taking the difference of the Navier-Stokes equations of unit mass density at two points $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\begin{aligned}
&\frac{\partial}{\partial t} w_i(\mathbf{x}_1,\mathbf{x}_2)+\left(u_j(\mathbf{x}_2)\frac{\partial}{\partial x_{2j}}+u_j(\mathbf{x}_1)\frac{\partial}{\partial x_{1j}}\right)w_i(\mathbf{x}_1,\mathbf{x}_2)\\
&\quad=-\left(\frac{\partial}{\partial x_{2i}}+\frac{\partial}{\partial x_{1i}}\right)\delta p(\mathbf{x}_1,\mathbf{x}_2)+\delta f_i(\mathbf{x}_1,\mathbf{x}_2)\\
&\qquad+\nu(\nabla_2^2+\nabla_1^2)w_i(\mathbf{x}_1,\mathbf{x}_2),
\end{aligned} \tag{2.2}$$

where δp and δf_i are the differences of pressure and external forces at two points

$$\begin{aligned}
\delta p(\mathbf{x}_1,\mathbf{x}_2)&=p(\mathbf{x}_2)-p(\mathbf{x}_1),\\
\delta f_i(\mathbf{x}_1,\mathbf{x}_2)&=f_i(\mathbf{x}_2)-f_i(\mathbf{x}_1).
\end{aligned} \tag{2.3}$$

Throughout the present paper we will employ the convention that the summation is taken over repeated indices of j and k without otherwise stated, while it is not taken over i. Introducing $\mathbf{X}$ and $\mathbf{r}$ as

$$\mathbf{X}=(\mathbf{x}_1+\mathbf{x}_2)/2, \quad \mathbf{r}=\mathbf{x}_2-\mathbf{x}_1, \tag{2.4}$$

Eq. (2.2) reduces to

$$\frac{\partial}{\partial t}w_i+V_j\frac{\partial}{\partial X_j}w_i+w_j\frac{\partial}{\partial r_j}w_i=-\frac{\partial}{\partial X_i}\delta p+\delta f_i+\nu\nabla_{\mathbf{X}}^2 w_i, \tag{2.5}$$

where $\mathbf{V}$ is the average velocity defined as

$$\mathbf{V}=[\mathbf{u}(\mathbf{x}_1)+\mathbf{u}(\mathbf{x}_2)]/2. \tag{2.6}$$

In deriving the right hand side of Eq. (2.5) we made use of the identity

$$(\nabla_2^2+\nabla_1^2)w_i=(\nabla_2+\nabla_1)^2 w_i=\nabla_{\mathbf{X}}^2 w_i, \tag{2.7}$$

because

$$\nabla_1\cdot\nabla_2 w_i=0.$$

The incompressible condition now takes the forms

$$\frac{\partial}{\partial r_j}w_j=\frac{\partial}{\partial X_j}w_j=\frac{\partial}{\partial r_j}V_j=\frac{\partial}{\partial X_j}V_j=0, \tag{2.8}$$

because

$$\frac{\partial}{\partial r_j}w_j=\frac{\partial}{\partial X_j}[u_j(\mathbf{X}+\mathbf{r}/2)-u_j(\mathbf{X}-\mathbf{r}/2)]=0$$

and so on. (Note that the partial derivative of w_i with respect to $\mathbf{x}_1$ should be carried out with $\mathbf{x}_2$ fixed and vice versa. Similarly the partial derivative of w_i with respect to $\mathbf{X}$ should be carried out with $\mathbf{r}$ fixed and vice versa.)

The pressure difference can be computed by taking the divergence of Eq. (2.5) with respect to $\mathbf{X}$, i.e., by operating $\partial/\partial X_i$ on Eq. (2.5). Then

$$\begin{aligned}
\nabla_{\mathbf{X}}^2\delta p&=-\frac{\partial}{\partial X_j}\left[V_k\frac{\partial}{\partial X_k}w_j+w_k\frac{\partial}{\partial r_k}w_j\right]\\
&=-2\frac{\partial}{\partial X_j}\left[w_k\frac{\partial}{\partial r_k}w_j\right],
\end{aligned} \tag{2.9}$$

because

$$\frac{\partial V_j}{\partial X_k}=\frac{\partial w_j}{\partial r_k}.$$

Hence the pressure gradient appearing in Eq. (2.5) becomes

$$\frac{\partial}{\partial X_i}\delta p(\mathbf{X},\mathbf{r})=-\int d\mathbf{X}' K_{ij}(\mathbf{X}-\mathbf{X}')w_k(\mathbf{X}',\mathbf{r})\times\frac{\partial}{\partial r_k}w_j(\mathbf{X}',\mathbf{r}), \tag{2.10}$$

where

$$K_{ij}(\mathbf{R})=\frac{1}{2\pi R^3}\left[\delta_{ij}-3\frac{R_iR_j}{R^2}\right]. \tag{2.11}$$

The equation for w_i^n is easily derived by multiplying Eq. (2.5) by nw_i^{n-1}:

$$\frac{\partial}{\partial t}w_i^n+V_j\frac{\partial}{\partial X_j}w_i^n+nw_i^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_i+\frac{\partial}{\partial X_i}\delta p\right)-nw_i^{n-1}\delta f_i =\nu\nabla_{\mathbf{X}}^2w_i^n-n(n-1)\nu w_i^{n-2}|\nabla_{\mathbf{X}}w_i|^2. \tag{2.12}$$

This type of equation was already introduced elsewhere [12,13,23,24]. The second term on the right-hand side of Eq. (2.12) represents the viscous dissipation of w_i^n stuff, while the first term does the spatial transport of the same stuff due to molecular viscosity, which conserves w_i^n in the entire system. It should be emphasized, therefore, that the first term on the right-hand side of Eq. (2.12) has nothing to do with the dissipation of w_i^n.

The right-hand side of (2.12) may be written in a different form:

$$\nu\left(\frac{1}{2}\nabla_{\mathbf{X}}^2+2\nabla_{\mathbf{r}}^2\right)w_i^n-n(n-1)\nu w_i^{n-2}[|\nabla_2w_i|^2+|\nabla_1w_i|^2] =\nu\left(\frac{1}{2}\nabla_{\mathbf{X}}^2+2\nabla_{\mathbf{r}}^2\right)w_i^n-n(n-1)\nu w_i^{n-2}\times[|\nabla_2u_i(\mathbf{x}_2)|^2+|\nabla_1u_i(\mathbf{x}_1)|^2], \tag{2.13}$$

where the content in the angular brackets is the sum of the energy dissipation rate at two points $\mathbf{x}_1$ and $\mathbf{x}_2$. However, the expression (2.12) is more meaningful in dealing with the velocity increment itself, because Eq. (2.12) indicates that the spatial gradient of w_i leads to the dissipation of w_i^n.

The equation for the nth order structure function is obtained by taking the ensemble average of Eq. (2.12). Since the system is stationary in time and homogeneous in space, the first term on the left-hand side of Eq. (2.12) does not contribute. The second term also vanishes because

$$\left\langle V_j\frac{\partial}{\partial X_j}w_i^n\right\rangle=\frac{\partial}{\partial X_j}\langle V_jw_i^n\rangle=0.$$

Then Eq. (2.12) reduces to

$$n\left\langle w_i^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_i+\frac{\partial}{\partial X_i}\delta p\right)\right\rangle-n\langle w_i^{n-1}\delta f_i\rangle =-n(n-1)\nu\langle w_i^{n-2}|\nabla_{\mathbf{X}}w_i|^2\rangle. \tag{2.14}$$

Here the last term on the left-hand side of Eq. (2.14) is the contribution from external forces, while the right-hand side of Eq. (2.14) is contributed by the molecular dissipation. If we focus on the universal region independent of how turbulence is excited externally, we can ignore the external forcing term

$$n\left\langle w_i^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_i+\frac{\partial}{\partial X_i}\delta p\right)\right\rangle =-n(n-1)\nu\langle w_i^n\rangle. \tag{2.15}$$

If we use Eq. (2.13) instead, we have another expression for $\langle w_i^n\rangle$ in place of Eq. (2.15):

$$n\left\langle w_i^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_i+\frac{\partial}{\partial X_i}\delta p\right)\right\rangle =2\nu\nabla_{\mathbf{r}}^2\langle w_i^n\rangle-2n(n-1)\langle w_i^{n-2}\varepsilon_i(\mathbf{x}_1,\mathbf{x}_2)\rangle, \tag{2.16}$$

where

$$\varepsilon_i(\mathbf{x}_1,\mathbf{x}_2)=\frac{\nu}{2}(|\nabla u_i(\mathbf{x}_2)|^2+|\nabla u_i(\mathbf{x}_1)|^2). \tag{2.17}$$

Note that $\varepsilon_i(\mathbf{x}_1,\mathbf{x}_2)$ is related to the average energy dissipation rate $\overline{\varepsilon}$ through

$$\overline{\varepsilon}=\sum_i\langle\varepsilon_i(\mathbf{x}_1,\mathbf{x}_2)\rangle. \tag{2.18}$$

The right-hand side of Eq. (2.16) consists of two terms. The first term represents the rate of molecular diffusion of $\langle w_i^n\rangle$ in r space, while the second term stands for the contribution due to the correlation between the energy dissipation rate and w_i^{n-2}. The molecular diffusion term is smaller by a factor of $(r_d/r)^{4/3}$ (r_d is the dissipation scale) than the inertial term within K41, so that we are led to

$$\left\langle w_i^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_i+\frac{\partial}{\partial X_i}\delta p\right)\right\rangle =-2(n-1)\langle w_i^{n-2}\varepsilon_{i(\mathbf{x}_1,\mathbf{x}_2)}\rangle \tag{2.19}$$

in the inertial region. If n is equated to 2 in Eq. (2.19), it reduces to the Kolmogorov's 4/5 law. Hence Eq. (2.19) may be called the generalized Kármán-Howarth-Kolmogorov relation.

In the following we focus only on the statistical behavior of the longitudinal increment. When $\mathbf{r}$ is chosen in the x direction, Eq. (2.15) with $i=1$ becomes

$$\left\langle w_1^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_1+\frac{\partial}{\partial X_1}\delta p\right)\right\rangle=-(n-1)\nu\langle w_1^{n-2}|\nabla_{\mathbf{X}}w_1|^2\rangle, \tag{2.20}$$

which is valid in the universal region including the dissipative one. On the other hand, in the inertial region we have

$$\left\langle w_1^{n-1}\left(w_j \frac{\partial}{\partial r_j} w_1 + \frac{\partial}{\partial X_1}\,\delta p \right)\right\rangle = -2(n-1)\langle w_1^{n-2}\varepsilon_1(\mathbf{x}_1,\mathbf{x}_2)\rangle. \qquad (2.21)$$

Hereafter the term $w_j(\partial/\partial_{r_j})w_1+(\partial/\partial X_1)\,\delta p$ is called the inertial term for convenience, which consists of the convection term plus the pressure term. We will address the following issues: (i) what a role the pressure gradient plays compared with the convection term and (ii) whether the correlation of the longitudinal dissipative rate $\varepsilon_1(\mathbf{x}_1,\mathbf{x}_2)$ with w_1^{n-2} affects the $(n+1)$th order structure function.

III. CONDITIONAL AVERAGE OF CONVECTION AND PRESSURE TERMS

In this section we focus on the roles of the convection term and the pressure gradient term on the left-hand side of Eq. (2.20). In particular we consider a problem on how the pressure term affects the convection term, to modify the inertial term. It is intuitively believed that the incompressible condition forces the pressure to screen the convection.

Instead of directly treating the left-hand side of Eq. (2.20), it is useful to rewrite it as

$$\int dw_1[A(w_1)-B(w_1)]w_1^{n-1}P(w_1), \qquad (3.1)$$

where $A(w_1)$ and $B(w_1)$ are the conditional averages of the convection term and the pressure gradient term with the value of the longitudinal velocity increment fixed at w_1;

$$A(w_1)=\left\langle w_j \frac{\partial}{\partial r_j} w_1 | w_1 \right\rangle \qquad (3.2)$$

and

$$B(w_1)=-\left\langle \frac{\partial}{\partial X_1}\delta p | w_1 \right\rangle. \qquad (3.3)$$

In Eq. (3.1) $P(w_1)$ is a probability density function (PDF) of the longitudinal velocity increment. In what follows we discuss $A(w_1)$ and $B(w_1)$ separately.

A. $A(w_1)$

Figure 1 is a plot of $A(w_1)$ against w_1 divided by $\sqrt{\langle w_1^2\rangle}$. $A(w_1)$ was computed using the data of DNS, which was carried out on 1024^3 meshes, with $R_\lambda=380$. The sampling was done at 138 different times over 6.6 eddy turnover times; the inertial region is located between $80\leqslant r/\eta\leqslant 200$. The detail of the simulation is given in Refs. [8,25]. In order to show the scale dependence of the behavior of $A(w_1)$ clearly, we give the curves for $19\leqslant r/\eta\leqslant 304$ in Fig. 1(a), and those for $4.8\leqslant r/\eta\leqslant 19$ in Fig. 1(b) separately. From these figures we have the following summary. (i) $A(w_1)$ resembles a parabola, and the curvature decreases with the increase of the scale as long as the scale is larger than 19η. For r less than 19η, i.e., in the dissipative region, the curvature decreases on the contrary as r decreases. (ii) The parabola is asymmetric, i.e., it is larger on the negative side of w_1 than on the positive side of w_1. This tendency is the most significant at the smallest separation $r=4.8\eta$. (iii) The minimum of $A(w_1)$ is negative, located around the origin of w_1.

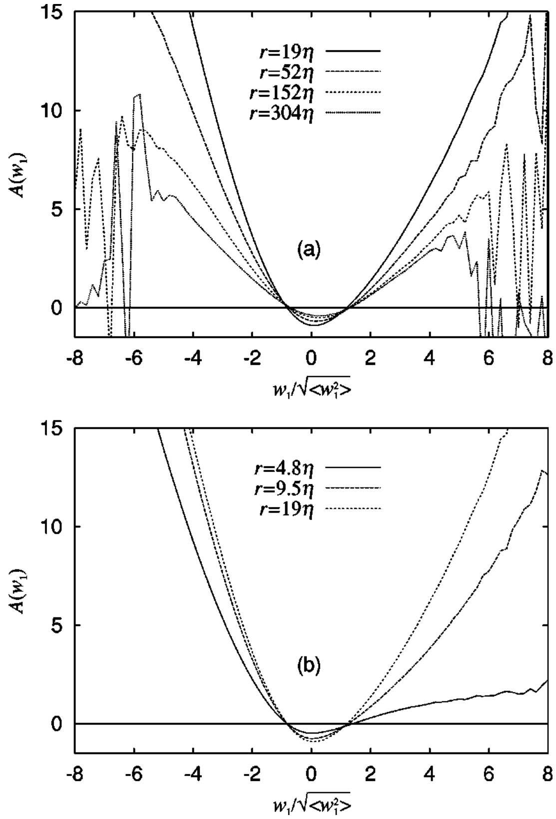

FIG. 1. Plot of $A(w_1)$ against $w_1/\sqrt{\langle w_1^2\rangle}$ (a) in the range between $r=19\eta$ and 304η and (b) in a range smaller than $r=19\eta$.

Let us reason the above features of $A(w_1)$. To this end, we introduce an illuminating model for w_i in Eq. (2.1). At first we are tempted to Taylor expand Eq. (2.1) in powers of r. However, the derivative of the velocity field is the strongest on a dissipative scale, so that one cannot directly apply the Taylor expansion to Eq. (2.1). Since w_i represents the intensity of the velocity field of a scale r, it makes no difference even if Eq. (2.1) is replaced by

$$w_i(\mathbf{x}_1,\mathbf{x}_2)=\tilde{u}_i(\mathbf{x}_2)-\tilde{u}_i(\mathbf{x}_1), \qquad (3.4)$$

where $\tilde{u}_i(\mathbf{x}_2)$ is the coarse-grained velocity field transmitted through a filter with width r; $\tilde{u}_i(\mathbf{x}_2)$ is smooth on scales smaller than r. Now we can Taylor expand Eq. (3.4):

$$w_i(\mathbf{X},\mathbf{r})=r_j\frac{\partial}{\partial X_j}\tilde{u}_i(\mathbf{X},\mathbf{r})=S_{ij}r_j-\frac{1}{2}\varepsilon_{ijk}r_j\Omega_k, \qquad (3.5)$$

where S_{ij} and Ω_k are slowly varying strain rate and vorticity of a scale r; they may be a function of $\mathbf{X}$ and $\mathbf{r}$, but we will ignore their dependence for simplicity.

Since $\mathbf{r}$ is assumed to be along a direction 1, r_1 is equated to r, while r_2 and r_3 go to zero. However the limiting procedure must be taken *only after the differentiation*. Let us write down w_i explicitly using Eq. (3.5):

$$w_1 = S_{11} r_1 + \left(S_{1\beta} - \frac{1}{2} \varepsilon_{1\beta k} \Omega_k \right) r_\beta , \tag{3.6}$$

$$w_\alpha = \left(S_{\alpha 1} - \frac{1}{2} \varepsilon_{\alpha 1 k} \Omega_k \right) r_1 + \left(S_{\alpha\beta} - \frac{1}{2} \varepsilon_{\alpha\beta k} \Omega_k \right) r_\beta , \tag{3.7}$$

where we employed the convention that Roman letters run over from 1 to 3, while Greek letters over 2 and 3 only.

Then the longitudinal convection term takes the form

$$w_j \frac{\partial}{\partial r_j} w_1 = w_1 \frac{\partial}{\partial r_1} w_1 + w_\alpha \frac{\partial}{\partial r_\alpha} w_1$$

$$\Rightarrow \left[S_{11}^2 + S_{1\alpha}^2 - \frac{1}{4} \Omega_\alpha^2 \right] r, \tag{3.8}$$

where the last expression was obtained after taking the limit that r_1 goes to r, while r_2 and r_3 go to zero. Since $A(w_1)$ is the conditional average of Eq. (3.8) with a fixed value of $w_1 = S_{11} r$, we have

$$A(w_1) = r \left\langle S_{11}^2 + S_{1\alpha}^2 - \frac{1}{4} \Omega_\alpha^2 | S_{11} \right\rangle . \tag{3.9}$$

First we tentatively assume that $S_{1\alpha}$ and Ω_α are statistically independent of S_{11}. Then

$$A(w_1) = S_{11}^2 r + \left\langle S_{1\alpha}^2 - \frac{1}{4} \Omega_\alpha^2 \right\rangle r. \tag{3.10}$$

The second term in Eq. (3.10) is equal to $-\langle S_{11}^2 \rangle$ because

$$\left\langle w_j \frac{\partial}{\partial r_j} w_1 \right\rangle = \left\langle w_j \frac{\partial}{\partial X_j} V_1 \right\rangle = \frac{\partial}{\partial X_j} \langle w_j \; V_1 \rangle = 0,$$

so that

$$r \left\langle S_{11}^2 + S_{1\alpha}^2 - \frac{1}{4} \Omega_\alpha^2 \right\rangle = 0. \tag{3.11}$$

Finally we are led to

$$A(w_1) = r[S_{11}^2 - \langle S_{11}^2 \rangle] = \frac{1}{r} [w_1^2 - \langle w_1^2 \rangle]. \tag{3.12}$$

The expression (3.12) is symmetric with respect to w_1, and agrees roughly with the observed form of $A(w_1)$ in Fig. 1 except for the asymmetricity in $A(w_1)$.

Next let us consider the asymmetric nature of $A(w_1)$. To this end we focus on the large amplitude of $w_1 = S_{11} r$. In the region where S_{11} is positive, the vorticity is amplified. Since the direction of the vorticity is, however, along the intermediate strain-rate eigenvector [26], Ω_α is large in (3.9). Hence the variable $S_{11}^2 - \frac{1}{4} \Omega_\alpha^2$ is much depressed from S_{11}^2 when w_1 is positive. In the region where S_{11} is negative, the enhancement of the vorticity does not operate, so that such a depression does not occur. In consequence, the asymmetric property of $A(w_1)$ is expected from Eq. (3.9).

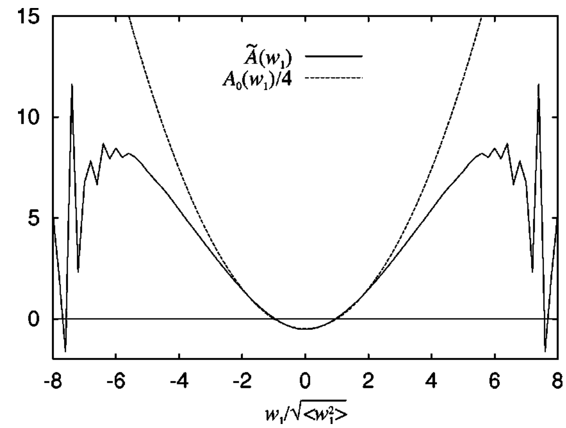

FIG. 2. Comparison of $A_0(w_1)/4$ with $\tilde{A}(w_1)$ at $r = 107\eta$.

It is important to check whether the result obtained above agrees quantitatively with the observed value of $A(w_1)$ by substituting the computed coarsed grained variables S_{11}, $S_{1\alpha}$, and Ω_α into Eq. (3.9). This problem will be investigated in the future. Here we are content with comparing the isotropic form (3.12), which is referred to as $A_0(w_1)$, with the observed isotropic value

$$\tilde{A}(w_1) = \frac{1}{2} [A(w_1) + A(-w_1)]. \tag{3.13}$$

There is a quantitative difference between $A_0(w_1)$ and $\tilde{A}(w_1)$: $A_0(w_1)$ is overestimated. In Fig. 2 we compare $A_0(w_1)/4$, which is by a factor four smaller than $A_0(w_1)$, to $\tilde{A}(w_1)$ at $r = 107\eta$. The comparison suggests that the approximation used in deriving Eq. (3.12) is quantitatively wrong, i.e., there is strong statistical correlation between S_{11} and $S_{1\alpha}$ and Ω_α. Figure 2 indicates, furthermore, that the discrepancy between $A_0(w_1)$ and $\tilde{A}(w_1)$ increases with the intensity of w_1. There are three conceivable reasons for it. (i) The statistical correlation between S_{11} and $S_{1\alpha}$ and Ω_α increases with the magnitude of w_1. (ii) There is a saturational effect for large w_1 at such an inertial separation, because the events with large amplitude are suppressed due to the finite Reynolds number. (iii) The number of samples is not sufficient.

The same tendency is observed for other separations: (i) We need a multiplication factor ranging from 1/3 to 1/4 to compare $A_0(0)$ with $\tilde{A}(0)$. (ii) For large amplitude of w_1 the discrepancy between $A_0(w_1)$ and $\tilde{A}(w_1)$ increases with $|w_1|$.

Let us consider the rate of energy transfer due to the convection term in scale space. The rate becomes

$$w_1 w_j \frac{\partial}{\partial r_j} w_1 = S_{11} \left[S_{11}^2 + S_{1\alpha}^2 - \frac{1}{4} \Omega_\alpha^2 \right] r^2, \tag{3.14}$$

where Eqs. (3.6) and (3.8) have been used, although the statistical correlation between S_{11}, $S_{1\alpha}$, and Ω_α must be taken

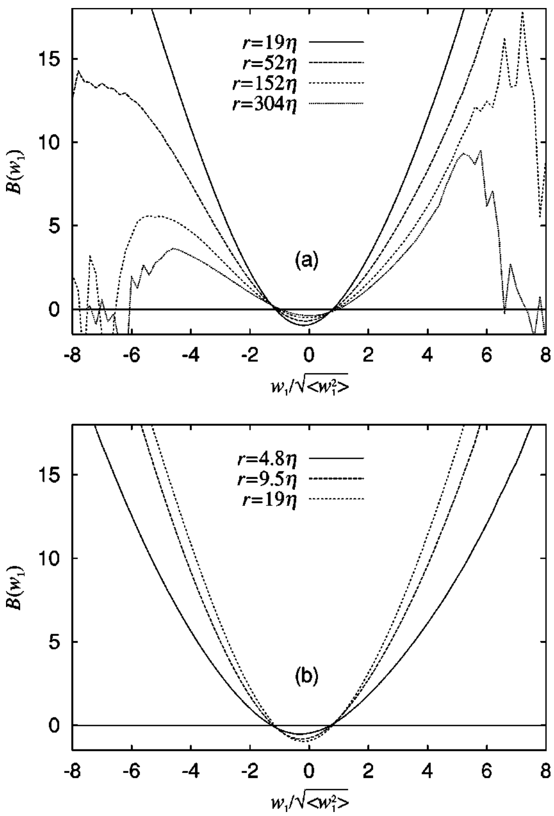

FIG. 3. Plot of $B(w_1)$ against $w_1/\sqrt{\langle w_1^2\rangle}$ (a) in the range between $r=19\eta$ and 304η and (b) in a range smaller than $r=19\eta$.

into consideration for more detailed quantitative evaluation. There are a few conclusions drawn from Eq. (3.14). (i) If one assumes that $S_{1\alpha}$ and Ω_α are statistically independent of S_{11}, we have the usual form for an average transfer rate because $\langle S_{11}\rangle=0$:

$$\left\langle w_1 w_j \frac{\partial}{\partial r_j} w_1 \right\rangle = \langle S_{11}^3\rangle r^2 = \frac{\langle w_1^3\rangle}{r}. \tag{3.15}$$

Since energy cascades toward smaller scales, $\langle w_1^3\rangle$ is negative. (ii) Equation (3.14) suggests that the energy cascades in an inverse direction in the region where the vorticity is strong in such a way that $\Omega_\alpha^2 > 4S_{1j}^2$ in consistent with the result by Horiuti [27].

Finally we consider the amplitude-dependent energy cascade rate. When $A(w_1)$ is given by Eq. (3.12), the amplitude-dependent energy transfer rate in r space is evaluated as

$$w_1 A(w_1) = \frac{1}{r} w_1 [w_1^2 - \langle w_1^2\rangle]. \tag{3.16}$$

Let us limit ourselves to $|w_1| > \sqrt{\langle w_1^2\rangle}$. In this case the transfer rate is negative for $w_1<0$, which means the energy cascade toward smaller scale. The situation for $w_1>0$ is opposite. This tendency is completely consistent with the Burgers' picture. For small amplitude such as $|w_1| < \sqrt{\langle w_1^2\rangle}$, on the other hand, the direction of the energy cascade is opposite to the case $|w_1| > \sqrt{\langle w_1^2\rangle}$.

B. $B(w_1)$

The computed value of $B(w_1)$ from the DNS is given in Fig. 3: the curves for $19 \leqslant r/\eta \leqslant 304$ in Fig. 3(a), and those for $4.8 \leqslant r/\eta \leqslant 19$ in Fig. 3(b). Most essential features of $B(w_1)$ are (i) $B(w_1)$ has the same sign as $A(w_1)$, (ii) the scale dependence of $B(w_1)$ is very similar to that of $A(w_1)$, and (iii) it is comparatively symmetric under $w_1 \leftrightarrow -w_1$ in contrast to $A(w_1)$. We will address these issues below.

Substituting Eq. (2.10) into Eq. (3.3) yields

$$B(w_1) = \int d\mathbf{R} K_{1j}(\mathbf{R}) \langle E_j(\mathbf{X}+\mathbf{R}) | w_1 ; \mathbf{X}\rangle, \tag{3.17}$$

where

$$E_j = w_k \frac{\partial}{\partial r_k} w_j \tag{3.18}$$

and

$$K_{1j}(\mathbf{R}) = \frac{1}{2\pi R^3}\left[\delta_{1j} - 3\frac{R_1 R_j}{R^2}\right]. \tag{3.19}$$

In Eq. (3.17) we have inserted the position vector $\mathbf{X}$ to signify that the average of $E_j(\mathbf{X}+\mathbf{R})$ is computed with a fixed value of w_1 at $\mathbf{X}$.

Let us rewrite Eq. (3.17) in the decomposed form

$$B(w_1) = \int d\mathbf{R} K_{11}(\mathbf{R}) \langle E_1(\mathbf{X}+\mathbf{R}) | w_1 ; \mathbf{X}\rangle + \int d\mathbf{R} K_{1\alpha}(\mathbf{R}) \langle E_\alpha(\mathbf{X}+\mathbf{R}) | w_1 ; \mathbf{X}\rangle. \tag{3.20}$$

The second term is the contribution from the transverse component E_α, where the kernel $K_{1\alpha}$ is proportional to $R_1 R_\alpha$. If turbulence has a peculiar structure such that $\langle E_1(\mathbf{X}+\mathbf{R}) | w_1;\mathbf{X}\rangle$ is an odd function of R_α, the nondiagonal component may contribute to Eq. (3.20). (If $\langle E_\alpha(\mathbf{X}+\mathbf{R}) | w_1;\mathbf{X}\rangle$ is an even function of R_1, it vanishes.) It is reasonable, however, to assume that the main contribution comes from the first term on the right-hand side of Eq. (3.20), whose kernel $K_{11}(\mathbf{R})$ is an even function of $\mathbf{R}$. Under this approximation

$$B(w_1) = \int d\mathbf{R} K_{11}(\mathbf{R}) \langle E_1(\mathbf{X}+\mathbf{R}) | w_1 ; \mathbf{X}\rangle, \tag{3.21}$$

where

$$K_{11}(\mathbf{R}) = \frac{1}{2\pi R^3}\left[1 - 3\frac{R_1^2}{R^2}\right]. \tag{3.22}$$

In order to relate $B(w_1)$ to $A(w_1)$ in a certain way we insert the state of w_1' at $\mathbf{X}+\mathbf{R}$ in the computation of the conditional average in the integrand of Eq. (3.21):

$$\begin{aligned}\langle E_1(\mathbf{X}+\mathbf{R})|w_1;\mathbf{X}\rangle\\ &=\int dw_1'\langle E_1(\mathbf{X}+\mathbf{R})|w_1';\mathbf{X}+\mathbf{R}\rangle C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})\\ &=\int dw_1' A(w_1')C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X}),\end{aligned}\tag{3.23}$$

where $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ is a probability that $w_1(\mathbf{X}+\mathbf{R})$ takes on a value w_1' provided that a value of $w_1(\mathbf{X})$ is specified at w_1. Hence C must satisfy a normalization condition

$$\int dw_1' C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})=1. \tag{3.24}$$

Note that the limiting form of $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ is such as

$$C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})=\begin{cases}\delta(w_1'-w_1) & \text{as } R\to 0\\ P(w_1') & \text{as } R\to\infty,\end{cases} \tag{3.25}$$

where $P(w_1')$ is the PDF of w_1'.

Plugging Eq. (3.23) into Eq. (3.21), we have

$$B(w_1)=\int dw_1' A(w_1')T(w_1'|w_1), \tag{3.26}$$

where

$$T(w_1'|w_1)=\int d\mathbf{R}K_{11}(\mathbf{R})C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X}). \tag{3.27}$$

If $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ is isotropic with respect to $\mathbf{R}$, the right-hand side of Eq. (3.27) vanishes due to the integration over the angle of $\mathbf{R}$. Hence the both limiting values of $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ in Eq. (3.25) do not make any contribution to the right hand-side of Eq. (3.27), so that the dominant contributing region of $\mathbf{R}$ is of the intermediate scale range, typically of order of r, as shown in Appendix A. Furthermore $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ must be anisotropic there.

Based on Eqs. (3.26) and (3.27), we propose a model which interprets the most essential property of $B(w_1)$ that it has the same sign as $A(w_1)$. The condition for Eq. (3.26) to be consistent with such a property is that $T(w_1'|w_1)$ is positive. Since $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ is positive definite, the integration over $\mathbf{R}$ in Eq. (3.27) should be dominant in the region where $K_{11}(\mathbf{R})$ is positive, i.e., the region where

$$R^2>3R_1^2\to R_\perp^2>2R_1^2, \tag{3.28}$$

where $R_\perp^2=R_2^2+R_3^2$; the shaded area in Fig. 4 corresponds to the area (3.28). Hence the integrated value of $K_{11}(\mathbf{R})C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ should be larger in the region (3.28) than in the other region.

To be more specific. Take a case where $|w_1|$ is large at $\mathbf{X}$, whose position is chosen at the orgin in Fig. 4. Let us estimate R_1^*, the spatial extension of $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ along direction 1 and $R_\perp^*$, that in perpendicular directions to x. Since $\mathbf{r}$ is along x direction $R_1^*\sim r$. How about a size of the

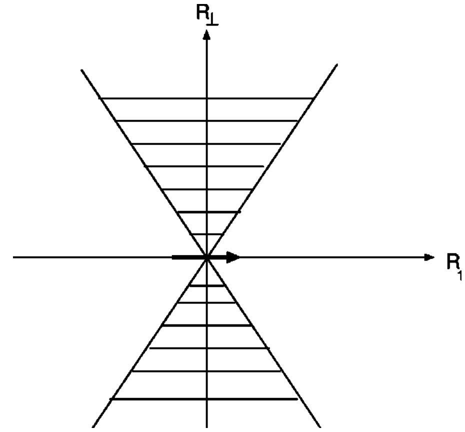

FIG. 4. In the shaded region the kernel K_{11} is positive. The boundary is determined by $R_\perp^2=2R_1^2$. The arrow at the origin represents $w_1(\mathbf{X},\mathbf{r})$.

perpendicular extension? If such an object is less localized in perpendicular directions, i.e., $R_\perp^*\gg R_1^*$, one can expect that $T(w_1'|w_1)$ is positive, yielding $B(w_1)$ with a value of the same sign as $A(w_1)$. A model is such that a Burgers-like compression occurs in the x direction, and this kind of structure extends in the perpendicular directions similar to a sheet.

Although the above model can interpret the feature that the sign of $B(w_1)$ is the same as that of $A(w_1)$, it also predicts that $B(w_1)\sim A(w_1)$ irrespective of sign of w_1 on the contrary to the observation that $B(w_1)$ is more symmetric under $w_1\leftrightarrow -w_1$ than $A(w_1)$. To overcome the discrepancy we expand the above model one more step; a localized region consists of a pair of the positive slope and negative slope, which is seen in Burgers turbulence. In the perpendicular directions to x such a configuration extends considerably as a sheet. In this model $C(w_1';\mathbf{X}+\mathbf{R}|w_1;\mathbf{X})$ is nonzero even if w_1' has an opposite sign to w_1. Namely,

$$\begin{aligned}B(w_1)=&\int_{w_1'>0} dw_1'\ A(w_1')T(w_1'|w_1)\\ &+\int_{w_1'<0} dw_1'\ A(w_1')T(w_1'|w_1).\end{aligned}\tag{3.29}$$

Equation (3.29) shows that $B(w_1)$ is contributed to by the positive component of $A(w_1)$ as well as the negative one, so that $B(w_1)$ is more symmetric than $A(w_1)$. It should be emphasized that the proposed model is not the unique model to interpret the properties of $B(w_1)$, but that this model is consistent with those porperties.

IV. RELATIVE MAGNITUDE OF *A* AND *B*

In the previous section the DNS-based analysis of $A(w_1)$ and $B(w_1)$ was separately given together with the theoretical interpretations. In this section we focus on the combined form of A and B. Making use of the definitions (3.2) and (3.3), the cascade rate of w_1 of arbitrary power in the universal range is given from Eq. (2.20) as

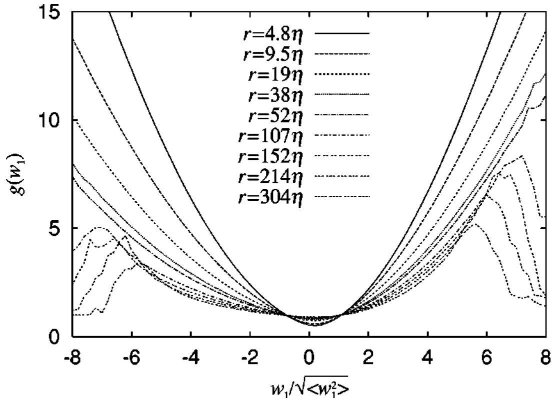

FIG. 5. Plot of $g(w_1)$ against $w_1/\sqrt{\langle w_1^2\rangle}$. $g(w_1)$ is related to $G(w_1)$ via $g(w_1)=G(w_1)/\nu\langle|\nabla_{\mathbf{x}}w_1|^2\rangle$.

$$\langle w_1^{n-1}H(w_1)\rangle=-(n-1)\nu\langle w_1^{n-2}G(w_1)\rangle, \tag{4.1}$$

where

$$H(w_1)=A(w_1)-B(w_1). \tag{4.2}$$

Equation (4.2) indicates that $H(w_1)$, i.e., the relative magnitude of $A(w_1)$ and $B(w_1)$, determines the rate of cascade of w^n stuff, which is balanced by the dissipation rate on the right-hand side of Eq. (4.1). In order to tackle this issue, we rewrite Eq. (4.1) as

$$\langle w_1^{n-1}H(w_1)\rangle=-(n-1)\langle w_1^{n-2}G(w_1)\rangle, \tag{4.3}$$

where $G(w_1)$ on the right hand side is another conditional average

$$G(w_1)=\nu\langle|\nabla\mathbf{x}w_1|^2|w_1\rangle. \tag{4.4}$$

Hence the information of $G(w_1)$ determines the relative magnitude of $A(w_1)$ and $B(w_1)$. $G(w_1)$ was already introduced elsewhere [28].

Figure 5 is $G(w_1)$ numerically computed from the DNS, where a normalized quantity $g(w_1)=G(w_1)/\nu\langle|\nabla_{\mathbf{x}}w_1|^2\rangle$ is displayed. Useful properties of $G(w_1)$ are listed as follows. $G(w_1)$ is positive definite as seen from Eq. (4.4). It increases with $|w_1|$ as a parabola as in Fig. 5, and the curvature decreases with the scale. Since the PDF of the turbulence is skewed toward negative increment, $P(w_1)G(w_1)$ is negatively skewed, i.e., it takes larger value at $w_1<0$ than at $w_1>0$ as appreciated in Fig. 6, which was also computed from the same DNS data. With the help of these properties the right-hand side of Eq. (4.3) is evaluated.

(i) For even n. Since w_1^{n-2} is positive irrespective of the sign of w_1, the right-hand side of Eq. (4.3) is *negative with large absolute value*.

(ii) For odd n. Since w_1^{n-2} changes sign for $w_1<0$ and for $w_1>0$, the cancellation occurs in the evaluation of the right-hand side of Eq. (4.3). The negatively skewed conditional average of $P(w_1)G(w_1)$ predicts that $\langle w_1^{n-2}|G(w_1)\rangle$ is of small magnitude with negative sign. Hence the right-hand side of Eq. (4.3) is *small positive*.

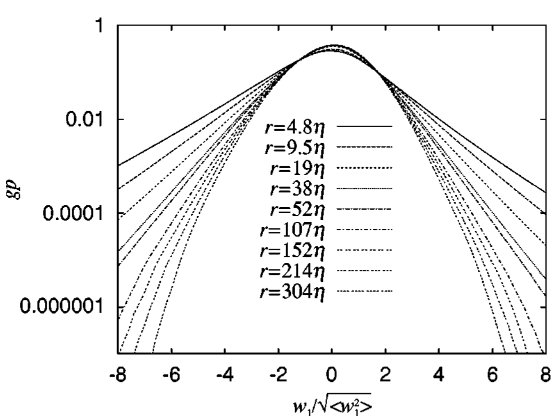

FIG. 6. Plot of $P(w_1)g(w_1)$ against $w_1/\sqrt{\langle w_1^2\rangle}$.

We demand $H(w_1)$ on the left-hand side of Eq. (4.3) to satisfy the numerical estimate of the right-hand side of Eq. (4.3). The left-hand side is computed for even n as well as for odd n.

(i) For even n we express the left-hand side in the following decomposed form:

$$\langle w_1^{n-1}H(w_1)\rangle=\langle|w_1|^{n-1}H_+(w_1)\rangle_+ -\langle|w_1|^{n-1}H_-(w_1)\rangle_-, \tag{4.5}$$

where a plus sign following H denotes that it is defined at $w_1>0$, and the same sign after the bracket means the average over the positive component of w_1, while a minus sign stands for the same kind of notation.

(ii) For odd n we have

$$\langle w_1^{n-1}H(w_1)\rangle=\langle|w_1|^{n-1}H_+(w_1)\rangle_+ +\langle|w_1|^{n-1}H_-(w_1)\rangle_-. \tag{4.6}$$

In order that the left- and right-hand sides of Eq. (4.3) agree with each other, Eq. (4.5) must be of large magnitude with negative sign, while Eq. (4.6) must be of small magnitude with positive sign. Hence we are led to

$$\langle|w_1|^{n-1}H_-(w_1)\rangle_->0, \tag{4.7}$$

$$\langle|w_1|^{n-1}H_+(w_1)\rangle_+<0, \tag{4.8}$$

and

$$|\langle|w_1|^{n-1}H_-(w_1)\rangle_-|>|\langle|w_1|^{n-1}H_+(w_1)\rangle_+|. \tag{4.9}$$

Based on the inequalities (4.7)–(4.9) we draw the following conclusions: from Eq. (4.7)

$$H_-(w_1)>0\to A_-(w_1)>B_-(w_1) \quad \text{for } w_1<0 \tag{4.10}$$

and from Eq. (4.8)

$$H_+(w_1)<0\to B_+(w_1)>A_+(w_1) \quad \text{for } w_1>0. \tag{4.11}$$

The inequalities (4.10) and (4.11) agree with the numerically calculated value of $H(w_1)$ from the DNS data as de-

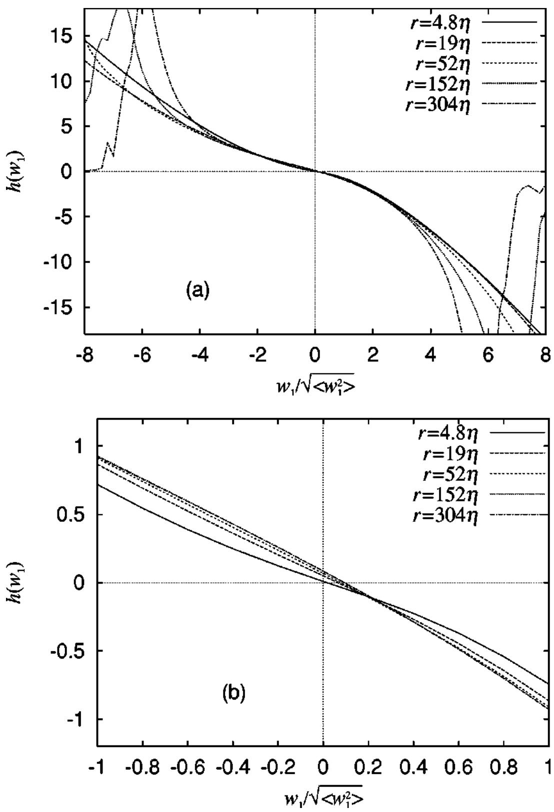

FIG. 7. Plot of $h(w_1)$ against $w_1/\sqrt{\langle w_1^2\rangle}$. $h(w_1)$ is related to $H(w_1)$ via $h(w_1)=\sqrt{\langle w_1^2\rangle}H(w_1)/\nu\langle|\nabla \mathbf{x} w_1|^2\rangle$. (a) In the wide range of w_1 and (b) in the vicinity of $w_1=0$.

picted in Fig. 7(a), where a normalized quantity $h(w_1) =\sqrt{\langle w_2^1\rangle}H(w_1)/\nu\langle|\nabla \mathbf{x} w_1|^2\rangle$ is displayed. The inequality (4.9), on the other hand, suggests that $H_-(w_1)$ is more significant than $H_+(w_1)$.

If we write $A_+(w_1)$, $A_-(w_1)$, $B_+(w_1)$, $B_-(w_1)$ in the order of the degree of importance

$$A_- > B_+ ,\quad B_- > A_+ , \tag{4.12}$$

where we cannot order B_+ and B_-. Equation (4.12) indicates that $A_-(w_1)$ is the most essential in driving fluctuations toward smaller scales in three-dimensional turbulence, while $A_+(w_1)$ is the least important. Both pressure terms $B_-(w_1)$ and $B_+(w_1)$ play an intermediate role between $A_-(w_1)$ and $A_+(w_1)$. Such a property in the pressure term is consistent with the conclusion in the preceding section that the pressure term is the average of the negative and positive convection terms.

Turn to the calculation of the energy transfer rate in r space due to the inertial term (convective plus pressure terms). The transfer rate is expressed as

$$w_1 H(w_1). \tag{4.13}$$

As can be seen from Fig. 7(a), $H(w_1)>0$ for $w_1<0$ and $H(w_1)<0$ for $w_1>0$, so that $w_1H(w_1)$ is negative in the entire region of w_1 irrespective of its sign. The energy cascades toward smaller scale for any w_1. Of interest is the behavior of $H(w_1)$ in the vicinity of $w_1=0$. Figure 7(b) is the closeup of $H(w_1)$; all the curves corresponding to the inertial range scale do not cross the origin. Although crossing points depend on the scale, they are roughly $w_1/\sqrt{\langle w_1^2\rangle} \approx 0.1$. This means that the inverse cascade of energy takes place in the interval $0 \leqslant w_1/\sqrt{\langle w_1^2\rangle} \leqslant 0.1$. The reason for it is not known to us at present.

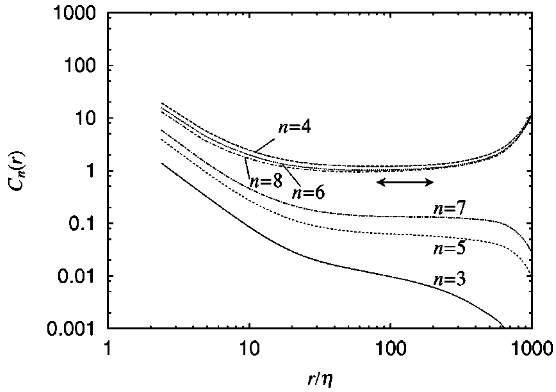

FIG. 8. Plot of $C_n(r)$ against r/η. The region between the two arrows is the inertial region.

Finally note that $H(w_1)$ is related to the conditional average of the Laplacian of w_1. If one takes the conditional average of Eq. (2.5) in the universal region, one has

$$\left\langle w_j \frac{\partial}{\partial r_j} w_1 + \frac{\partial}{\partial X_1}\delta p \middle| w_1 \right\rangle = \nu \langle \nabla^2 \mathbf{x} w_1 | w_1\rangle. \tag{4.14}$$

Hence

$$H(w_1) = \nu \langle \nabla^2 \mathbf{x} w_1 | w_1\rangle. \tag{4.15}$$

This definition of the conditional average was introduced elsewhere [28].

V. RELATIONSHIP BETWEEN INERTIAL TERM AND DISSIPATION RATE

In this section we will investigate what role the dissipation rate, i.e., the right-hand side of Eq. (2.21), plays against the inertial term. To this end, we computed

$$C_n(r) = -\frac{2n(n-1)\langle \varepsilon_1(\mathbf{x}_1,\mathbf{x}_2) w_1^{n-2}\rangle}{\langle w_1^{n+1}\rangle / r} \tag{5.1}$$

for various integers of n against r using the data of DNS, as depicted in Fig. 8, where the inserted solid line with arrows at ends $r/\eta = 80$ and 200 stands for the inertial region. We summarize the essential points drawn from Fig. 8. (i) For $n \geqslant 4$ $C_n(r)$ takes constant value C_n^* independent of r in the inertial region. [$C_5(r)$ depends slightly on r, but it is regarded as independent of r.] It is certain that $C_3(r)$ is not constant in the inertial region. (ii) For even integers of n C_n^* seem to approach a constant value A from above as $C_4^* > C_6^* > C_8^*$ as n increases. (iii) For odd n C_n^* increases with n. At the level of present resolution we could compute only up to C_7. It is clear that $C_7^* > C_5^*$. At this moment it is not confirmed that C_n^* asymptotically approaches A for odd n. It is probable that C_n with odd n converges to a different value.

We conclude that $C_n(r)$ approach constant values for large n in the inertial region. What does this mean? To answer to it, let us focus on Eq. (2.21), which is valid in the inertial range. As shown in Appendix A the pressure gradient term behaves in a similar way to the convection term, so that one can express the inertial term as

$$\frac{\partial}{\partial r_j}\langle w_j w_1^n\rangle + n\left\langle w_1^{n-1}\frac{\partial}{\partial X_1}\delta p\right\rangle = D(n)\frac{\partial}{\partial r_j}\langle w_j w_1^n\rangle. \tag{5.2}$$

Hence Eq. (2.21) reduces to

$$D(n)\frac{\partial}{\partial r_j}\langle w_j w_1^n\rangle = -2n(n-1)\ \langle w_1^{n-2}\varepsilon_1(\mathbf{x}_1,\mathbf{x}_2)\rangle, \tag{5.3}$$

where the following relation has been used:

$$\left\langle w_1^{n-1} w_j \frac{\partial}{\partial r_j} w_1\right\rangle = \frac{1}{n}\left\langle w_j \frac{\partial}{\partial r_j} w_1^n\right\rangle = \frac{1}{n}\frac{\partial}{\partial r_j}\langle w_j w_1^n\rangle.$$

For large n the approximation

$$\frac{\partial}{\partial r_j}\langle w_j w_1^n\rangle \approx \frac{\partial}{\partial r}\langle w_1^{n+1}\rangle = \frac{\zeta_{n+1}}{r}\langle w_1^{n+1}\rangle \tag{5.4}$$

holds in the inertial region, where ζ_{n+1} is the scaling exponent of the structure function of $\langle w_1^{n+1}\rangle$. Substituting Eq. (5.4) into Eq. (5.3) and combining the result with Eq. (5.1), we have

$$C_n(r) = \zeta_{n+1} D(n). \tag{5.5}$$

Substituting $\zeta_n \propto n$ [29] yields

$$D(n) = O\left(\frac{1}{n}\right). \tag{5.6}$$

Equation (5.6) indicates that the pressure screens the convection term considerably, but not in a perfect way.

Another interesting point found in Fig. 8 is the observation that $C_3(r)$ is not constant even in the inertial region. Putting n equal to 3 in Eq. (2.21), we have

$$\frac{1}{3}\frac{\partial}{\partial r_j}\langle w_j w_1^3\rangle + \left\langle w_1^2\frac{\partial}{\partial X_1}\delta p\right\rangle = -4\langle \varepsilon_1(\mathbf{x}_1,\mathbf{x}_2) w_1\rangle. \tag{5.7}$$

A plot of $\langle \epsilon_1(\mathbf{x}_1,\mathbf{x}_2)w_1\rangle r/\langle w_1^4\rangle$ in Fig. 8 indicates that it decreases with r. It means that the dissipation term is irrelevant in the inertial region. Namely,

$$\frac{1}{3}\frac{\partial}{\partial r_j}\langle w_j w_1^3\rangle + \left\langle w_1^2\frac{\partial}{\partial X_1}\delta p\right\rangle = 0 \tag{5.8}$$

holds in the inertial region. If Eq. (5.8) is rewritten,

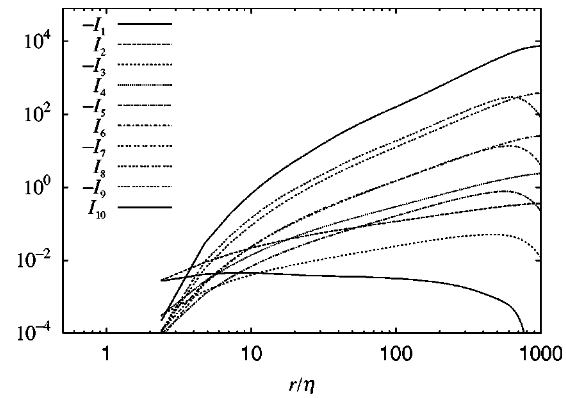

FIG. 9. Plot of I_n against r/η. The curves at $r/\eta = 200$ are I_1, I_3, I_2, I_5, I_4, I_6, I_7, I_8, I_9, and I_{10} upward from the bottom.

$$\left\langle w_1^2\left(\frac{\partial}{\partial r_j} w_j w_1 + \frac{\partial}{\partial X_1}\delta p\right)\right\rangle = 0.$$

At the level of fourth order the pressure almost screens the convection term

$$\frac{\partial}{\partial r_j} w_j w_1 + \frac{\partial}{\partial X_1}\delta p \approx 0, \tag{5.9}$$

and the effect of fluctuating dissipation rate does not come in. At order of three it is well known that the structure function is not affected by the fluctuating dissipation rate. For higher n the scaling exponents are affected by it. At present we do not reason why the fourth order structure function is not affected by the intermittency effect of the dissipation field.

VI. ROLE OF DISSIPATIVE STRUCTURE

In the preceding section we showed that the dissipative structure plays a crucial role in the structure functions of order higher than five. On the other hand, as far as the fourth order structure function is concerned, the dissipation term is shown to be irrelevant. In this context it is of interest to investigate the correlation of dissipation rate $\varepsilon_1(\mathbf{x}_1, \mathbf{x}_2)$ with $[w_1(\mathbf{X},\mathbf{r})]^n$, i.e.,

$$I_n \equiv \langle w_1^n \varepsilon_1\rangle. \tag{6.1}$$

Other interesting quantities are introduced by extending n into noninteger. For that purpose one can imagine two types of correlation

$$J_n \equiv \langle |w_1|^n \varepsilon_1\rangle \tag{6.2}$$

and

$$K_n \equiv \langle \mathrm{sgn}(w_1)|w_1|^n \varepsilon_1\rangle, \tag{6.3}$$

where $\mathrm{sgn}(x) = 1$ when $x > 0$ and -1 when $x < 0$. In the following we deal only with I_n and J_n in detail.

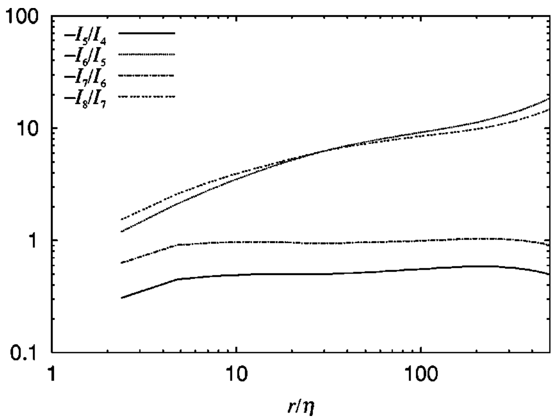

FIG. 10. Comparison of I_{2n+1}/I_{2n} to I_{2n+2}/I_{2n+1}.

A. I_n

In this subsection we will show how $\varepsilon_1(\mathbf{x}_1,\mathbf{x}_2)$ is correlated to $[w_1(\mathbf{X},\mathbf{r})]^n$. Figure 9 is a plot of I_n against r/η for n from 1 to 10 based on the DNS. The inertial range is located in between $r/\eta=80$ and 200. The results are summarized as follows. (i) For $n\geqslant 2$ I_n scales with r in a power law in the inertial region, although the slope is less than the K41 value $n/3$, reflecting the intermittency effects. It is well balanced with the left-hand side of Eq. (2.21) as discussed in Sec. V. (ii) Perplexing is the case $n=1$. I_1 decreases with r, on the contrary to the expectation $I_1\sim r^{1/3}$. This tendency is confirmed for many simulations with Reynolds numbers less than 380 together with Jet data of $R_\lambda=380$ [30], although those results are not shown here. (iii) In Fig. 9 we notice that the slope of I_6 is very close to that of I_7. A similar tendency holds between I_8 and I_9. To illuminate such a tendency, we compare I_{2n+1}/I_{2n} with I_{2n+2}/I_{2n+1}. The comparison is depicted in Fig. 10, which indicates that the former is almost independent of r in the inertial region, while the latter is an increasing function of r.

The property (iii) is understandable. I_n can be expressed in terms of the conditional average of ε_1 as

$$I_n=\int \langle \varepsilon_1|w_1\rangle w_1^n P(w_1)dw_1. \tag{6.4}$$

Since $\langle \varepsilon_1|w_1\rangle \approx G(w_1)$ in the inertial region, I_n is equal to

$$I_n=\int w_1^n G(w_1)P(w_1)dw_1. \tag{6.5}$$

Since $G(w_1)P(w_1)$ is negatively skewed for large amplitude of w_1 as seen in Fig. 6, it is very probable that the resemblance of I_{2n+1} to I_{2n} is stronger than that of I_{2n+1} to I_{2n+2}. This property combined with Eq. (5.3) suggests that $\langle w_1^{2m}\rangle$ is closer to $\langle w_1^{2m-1}\rangle$ than to $\langle w_1^{2m+1}\rangle$, in accordance with the observation by Stolovitzky, Sreenivasan, and Juneja [22]; see also Ref. [31].

B. J_n

Turn to the correlation of the longitudinal dissipation rate ε_1 with the absolute value of the velocity increment. Figure 11 is a plot of J_n against r/η for n from 1 to 10. We can see that J_n scales in a logarithmically ordered way with n in contrast to I_n.

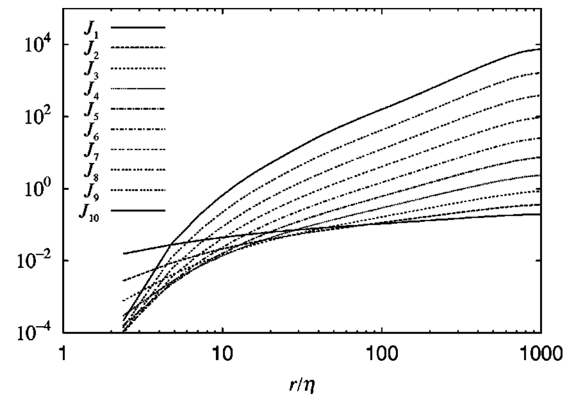

FIG. 11. Plot of J_n against r/η. The curves represent J_1 to J_{10} at $r/\eta=200$ from bottom to top.

What is the significance of the this result? Equation (2.12) leads us to

$$\langle |w_1|^{n-1}H(w_1)\rangle_+ - \langle |w_1|^{n-1}H(w_1)\rangle_- = -\nu(n-1)\langle |w_1|^{n-2}|\nabla \mathbf{x} w_1|^2\rangle, \tag{6.6}$$

which was derived in Appendix B. The right-hand side of Eq. (6.6) is nothing but Eq. (6.2) in the inertial region. According to the arguments in Sec. IV $H(w_1)$ is positive for $w_1<0$ and negative for $w_1>0$, so that both terms on the left-hand side of Eq. (6.6) are negative, bringing about no cancellation. One can approximate the left-hand side of Eq. (6.6) by

$$-\alpha D(n)\frac{\langle |w_1|^{n+1}\rangle}{r}, \tag{6.7}$$

in accordance with Eq.(5.2). Here α is a certain numerical constant independent of n and $D(n)$ is of order of $1/n$. Then Eq. (6.6) becomes

$$\alpha D(n)\frac{\partial}{\partial r}\langle |w_1|^{n+1}\rangle = 2n(n-1)J_{n-2}. \tag{6.8}$$

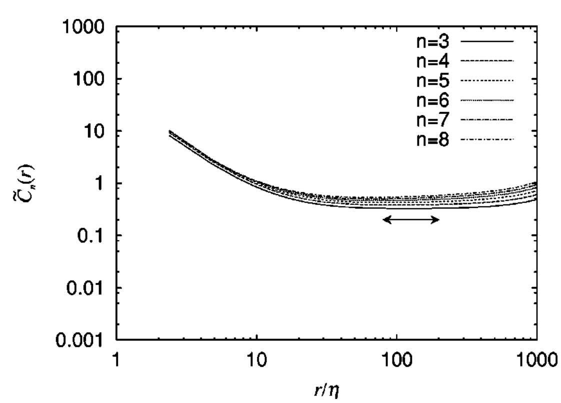

FIG. 12. Plot of $\tilde{C}_n(r)$ against r/η.

Equation (6.8) states that J_{n-2} is related to the $(n+1)$th order structure function of the absolute value of the velocity difference.

Now we are ready to show that the expression (6.8) is correct by plotting

$$\tilde{C}_n(r)=\frac{2n(n-1)J_{n-2}}{\langle|w_1|^{n+1}\rangle/r} \tag{6.9}$$

against r/η in Fig. 12, where $\tilde{C}_n(r)$ takes a constant value for any n. It seems that it approaches an asymptotic value systematically in the inertial region. Therefore the scaling exponents of the structure function of the absolute value of the longitudinal velocity difference orderly increases with n in accordance with the previous result in Ref. [8].

VII. COMPARISON WITH MEAN FIELD THEORY

In this section we wish to discuss the obtained results in comparison with the mean field approximation [12,13]. It may be useful to summarize the theory by Yakhot [12]. The equation for the structure function contains the convection term, the pressure term, and the dissipation term. If the pressure and dissipation terms are expressed in terms of the structure functions of other orders through the introduction of the conditional averages, the equation is closed, so that structure functions of different orders are related to each other, from which the scaling exponents will be derived. He started with the equation without the pressure and dissipation terms, and then, include those effects in a perturbative way. This approximation was expected to be valid at the dimension d close to d_c, which is the critical dimension which distinguishes the three-dimensional turbulence from the two-dimensional one. Whether $d=3$ is enough close to d_c must be determined by the comparison with experiment and DNS.

Kurien and Sreenivasan [13] investigated the validity of those approximations based on the data in boundary layer turbulence with $R_\lambda=10\,700$. Their strategy is to evaluate the contribution of the pressure term, which cannot be measured in experiment, by analyzing the equation for structure function of even orders where the effect of the dissipation term is considered to be small. They began with Eq. (2.21) with odd n, where the dissipation term is ignored:

$$\left\langle w_1^{2n}\left(w_j\frac{\partial}{\partial r_j}w_1+\frac{\partial}{\partial X_1}\delta p\right)\right\rangle=0. \tag{7.1}$$

Substituting the measured contribution of the convection term into Eq. (7.1) enabled them to estimate the contribution of the pressure term as 10% of the component of the convection term such as $\langle w_1^{2n+1}\partial w_1/\partial r\rangle$. Then they turned to the equation for the structure function of odd order; plugging even n into Eq. (2.21) yields

$$\left\langle w_1^{2n+1}\left(w_j\frac{\partial}{\partial r_j}w_1+\frac{\partial}{\partial X_1}\delta p\right)\right\rangle$$
$$=-2(2n+1)\langle w_1^{2n}\varepsilon_1(\mathbf{x}_1,\mathbf{x}_2)\rangle. \tag{7.2}$$

Substituting a model expression for the pressure, which was constructed using the result for the even order structure function, they evaluated the left-hand side of Eq. (7.2), which must be balanced with the right-hand side. The result is that the dissipation term dominates the pressure term; the dissipation term balances about 85 to 90% of the convection term.

In the present paper we investigated the full equation. We did not decompose the convection term in Eq. (7.1) into the sum of $\langle w_1^{2n+2}\rangle$ and $\langle w_1^{2n}w_2^2\rangle$ because the longitudinal convection term $w_k(\partial/\partial r_k)w_1$ is dealt with as a whole in the form of $A(w_1)$. Hence we cannot comment on how large the pressure term is as compared with the decomposed convection terms.

Some conclusions obtained in the present paper can be compared with the mean field approximation. At the level of fourth order, the effect of the dissipation term turns out to be negligible in the inertial region, which agrees with the assumption employed in the mean field approximation. In the equations for the structure functions of even order higher than 6, however, the dissipation term cannot be negligible, which balances the inertial term (convective plus pressure terms) in a scaling sense.

Finally we want to add the following remark. Mean field theory [12,13] assumed that I_n is small for odd n in the inertial region. According to Fig. 9 this assumption does not look so good, since I_n with odd n is only a little smaller than I_n with even n. The comparison of I_n with the inertial term, as done in Fig. 8, however, indicates that the contribution from odd n is smaller by a factor 10 than that from even n in agreement with the assumption in mean field theory.

VIII. DISCUSSION

In this section we will argue the origin of the anomalous scaling in three-dimensional turbulence based on the results obtained in this paper. Since we have shown elsewhere [8] that the scaling exponents of the longitudinal structure functions computed from our DNS data are agreement with the currently accepted values, the detailed discussion of the scaling exponents is not given here.

Instead the following issues are discussed. What is the cause of the anomalous scaling? Is the dissipative structure responsible for it? Does the homogeneous integrodifferential equation without the dissipative term, i.e.,

$$\left\langle w_1^{n-1}w_j\frac{\partial}{\partial r_j}w_1\right\rangle-\int d\mathbf{R}K_{1j}(\mathbf{R})\langle w_1^{n-1}(\mathbf{X},\mathbf{r})$$
$$\times w_k(\mathbf{X}+\mathbf{R},\mathbf{r})\frac{\partial}{\partial r_k}w_j(\mathbf{X}+\mathbf{R},\mathbf{r})\rangle=0, \tag{8.1}$$

yield a solution with the lower scaling exponents for the structure functions? What is a role of the pressure term in structure functions?

The study of $C_n(r)$ in Sec. V indicates that $\langle\varepsilon_1 w_1^n\rangle$ scales in the same way as $\langle w_1^{n+3}\rangle/r$ as long as $n\geqslant 2$. This implies that the dissipative structure is responsible for the anomalous scaling of the structure functions of order larger than 5. Since the pressure term scales as the convection term as shown in Appendix A, it contributes to the numerical coefficients of those structure functions, but not to their scalings. The homogeneous equation (8.1) cannot yield a more intermittent solution than the inhomogeneous solution.

The above discussion does not hold for the fourth order structure function, which scales anomalously as $r^{1.30}$ [8] compared with $r^{4/3}$ in K41. The equation for the fourth order structure function is confirmed not to be affected by the dissipative term, and the homogeneous integrodifferential equation must yields the anomalous scaling solution. It suggests that the pressure term may be responsible for the anomalous scaling of fourth order.

Hence the following scenario is possbile. At any order the pressure term brings about the anomalous scaling. The dissipative term only adjusts to the inertial term, i.e., the summation of the convective and pressure terms. At fourth order such a balance is broken, but at larger orders the balance is satisfied.

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APPENDIX A: CONTRIBUTION OF THE PRESSURE TERM IN EQ. (2.19)

Let us focus on the pressure term in Eq. (2.19). Substituting Eq. (2.10) into it, we have

$$\left\langle w_i^{n-1}\frac{\partial}{\partial X_i}\delta p\right\rangle = -\int d\mathbf{R} K_{ij}(\mathbf{R})L_{ij}(\mathbf{R},\mathbf{r}), \tag{A1}$$

where

$$L_{ij}(\mathbf{R},\mathbf{r}) = \langle w_i(\mathbf{X},\mathbf{r})^{n-1}E_j(\mathbf{X}+\mathbf{R},\mathbf{r})\rangle, \tag{A2}$$

where E_j is defined in Eq. (3.17). In Eq. (A1)

$$K_{ij}(\mathbf{R}) = \frac{1}{2\pi R^3}\left[\delta_{ij} - 3\frac{R_iR_j}{R^2}\right]. \tag{A3}$$

In Eq. (A1) the summation is taken only over j, not over i. Since $\mathbf{r}$ is chosen along x direction, the tensor form of $L_{ij}(\mathbf{R},\mathbf{r})$ is specified only by $\mathbf{R}$. Then $L_{ij}(\mathbf{R},\mathbf{r})$ can be represented as

$$L_{ij}(\mathbf{R},\mathbf{r}) = \langle w_i(\mathbf{X},\mathbf{r})^{n-1}E_j(\mathbf{X},\mathbf{r})\rangle f_{ij}(\mathbf{R}/r), \tag{A4}$$

where the summation is not taken over j. Then

$$\left\langle w_i^{n-1}\frac{\partial}{\partial X_i}\delta p\right\rangle = -\langle w_i(\mathbf{X},\mathbf{r})^{n-1}E_j(\mathbf{X},\mathbf{r})\rangle \times \int d\mathbf{R} K_{ij}(\mathbf{R}) f_{ij}(\mathbf{R}/r). \tag{A5}$$

Here the summation is taken over j. The first factor on the right-hand side of Eq. (A5)

$$\langle w_i(\mathbf{X},\mathbf{r})^{n-1}E_j(\mathbf{X},\mathbf{r})\rangle \tag{A6}$$

is expected to follow the same scaling as the convection term for $j=i$.

Now turn to the examination as to whether another scaling factor in r arises from the geometrical factor

$$\int d\mathbf{R} K_{ij}(\mathbf{R}) f_{ij}(\mathbf{R}/r). \tag{A7}$$

First we consider the contribution to Eq. (A7) from near the origin $R=0$. Since $f_{ij}(0)=1$, $f_{ij}(\mathbf{R})$ can be Taylor expanded as

$$f_{ij}(\mathbf{R}/r) = 1 + C_{ijk}R_k/r. \tag{A8}$$

When $f_{ij}=1$, Eq. (A7) becomes zero due to the angle integration of $\mathbf{R}$, so that only the second term in Eq. (A8) contributes to Eq. (A7); the contribution from $R_* + dR > R > R_*$ increases with R_*. Second consider the far region of R. For $R \geqslant r$, f_{ij} will decay, so that it is safely written as

$$f_{ij}(\mathbf{R}/r) \sim \left(\frac{R}{r}\right)^{-\delta}, \tag{A9}$$

where δ is positive. This time the contribution from $R_* + dR > R > R_*$ decreases with R_*. The above consideration indicates that the main contribution comes from $R \sim cr$, where c is of order of unity, which implies that Eq. (A7) is a numerical factor. Therefore the left-hand side of Eq. (A1) scales in exactly the same way as the convection term.

APPENDIX B: DERIVATION OF EQ. (6.6)

Let us start with Eq. (2.12),

$$\begin{aligned}\frac{\partial}{\partial t}w_1^n + \frac{\partial}{\partial X_j}V_j w_1^n + n w_1^{n-1} &\times\left(w_j\frac{\partial}{\partial r_j}w_1 + \frac{\partial}{\partial X_1}\delta p\right) - n w_1^{n-1}\delta f_1 \\ &= \nu\nabla_{\mathbf{X}}^2 w_1^n - n(n-1)\nu w_1^{n-2}|\nabla_{\mathbf{X}} w_1|^2,\end{aligned} \tag{B1}$$

and the equation obtained through multiplying Eq. (B1) by $(-1)^n$,

$$\frac{\partial}{\partial t}(-w_1)^n+\frac{\partial}{\partial X_j}V_j(-w_1)^n-n(-w_1)^{n-1}\times\left(w_j\frac{\partial}{\partial r_j}w_1+\frac{\partial}{\partial X_1}\delta p\right)+n(-w_1)^{n-1}\delta f_1=\nu\nabla_{\mathbf{X}}^2(-w_1)^n-n(n-1)\nu(-w_1)^{n-2}|\nabla_{\mathbf{X}}w_1|^2. \tag{B2}$$

Let us take the sum of the integration of Eq. (B1) multiplied by $P(w_1)dw_1$ from 0 to ∞ and that of Eq. (B2) multiplied by $P(w_1)dw_1$ from $-\infty$ to 0. The time derivative term becomes

$$\frac{\partial}{\partial t}\langle|w_1|^n\rangle,$$

which vanishes because of the stationarity of turbulence. The convection term

$$\frac{\partial}{\partial X_j}\langle V_j|w_1|^n\rangle$$

also vanishes because of the homogeneity of the system. The external forcing term can be neglected in the universal region. The viscous transport term

$$\nu\nabla_{\mathbf{X}}^2\langle|w_1|^n\rangle$$

becomes zero because of the homogeneity. Finally we have

$$\int_0^\infty w_1^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_1+\frac{\partial}{\partial X_1}\delta p\right)P(w_1)dw_1-\int_{-\infty}^0(-w_1)^{n-1}\left(w_j\frac{\partial}{\partial r_j}w_1+\frac{\partial}{\partial X_1}\delta p\right)P(w_1)dw_1=-\nu(n-1)\langle|w_1|^{n-2}|\nabla_{\mathbf{X}}w_1|^2\rangle. \tag{B3}$$

Rewriting Eq. (B3) with the use of $H(w_1)$, we have

$$\langle w_1^{n-1}H(w_1)\rangle_+-\langle|w_1|^{n-1}H(w_1)\rangle_-=-\nu(n-1)\langle|w_1|^{n-2}|\nabla\mathbf{x}w_1|^2\rangle. \tag{B4}$$

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